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DISKOSEISMOLOGY: PROBING ACCRETION DISKS. I. TRAPPED ADIABATIC OSCILLATIONS

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ABSTRACT

We analyze the normal modes of acoustic oscillations within thin accretion disks which are terminated by an innermost stable orbit around a slowly rotating black hole or weakly magnetized compact neutron star. The dominant relativistic effects, which allow modes to be trapped within the inner region of the disk, are approximated via a modified Newtonian potential. A general formalism is developed for investigating the adiabatic oscillations of arbitary unperturbed disk models. We explore the generic behavior via an expansion of the Lagrangian displacement about the plane of sysmmetry and by assuming separable solutions with the same radial wavelength for the horizontal and vertical perturbations. We obtain the lowest eigenfrequencies and eigenfunctions of a particular set of radial and quadrupole modes which have minimum motion normal to the plane. These modes correspond to the standard dispersion relation of disk theory. In future papers we will study a wide variety of modes and compute the rates of growth or damping of these modes due to gravitational radiation and various models of viscosity.

Subject headings: black holes — hydrodynamics — stars: accretion — wave motions

1. INTRODUCTION

Helioseismology (for a review see Deubner & Gough 1984) has given us the first detailed information about the interior of the Sun outside its central region (where the neutrinos are emitted), and its extension to observations of oscillations of other stars promises to also provide a powerful probe. In principle, the same approach can be used to explore the structure of accretion disks. We have initiated such a study in order to investigate what we might be able to learn about the nature of such accretion onto relativistic objects—from weakly magnetized neutron stars to massive black holes. We also hope to provide motivation for more sensitive, longer duration, and shorter time resolution observations of candidates for such systems.

Attention will be focused on oscillations which have a significant component in the plane of symmetry ($z = 0$) of the unperturbed disk, since we shall see that only such modes can be confined to the inner region of the disk, which is physically more interesting and observationally most luminous. The complementary case of "vertical" oscillations, which mainly reflect the local structure of the disk, has been studied by Cox (1981) and others. Carroll et al. (1985) have carried out a local analysis of three dimensional oscillations of purely Newtonian disks. However, the fact that general relativity forces the radial epicyclic frequency to reach a maximum outside the inner edge of a disk which is not terminated by a stellar surface or magnetosphere allows modes to be trapped in that region, as first pointed out by Kato & Fukue (1980). The spectrum of this cavity thus reflects the properties of relativistic gravity as well as the physics of the inner accretion disk.

In this initial exploratory phase of our investigation we shall neglect the gravitational field of the disk and approximate the dominant relativistic effects of the slowly rotating compact mass M via a modified Newtonian potential

$$\Phi = -(GM/r)[1 - 3(GM/rc^2) + 12(GM/rc^2)^2] . \tag{1.1}$$

Its innermost stable circular orbit is at $r = 6GM/c^2$, and it also has the correct general relativistic value of the angular velocity Ω (as measured at infinity) at that radius. This form of the potential yields values of both angular velocity Ω and radial epicylic frequency κ that are closer to the general relativistic values (for radii larger than $6GM/c^2$) than does the potential of Muchotrzeb & Paczyński (1982), for instance. A comparison of these quantities can be found in Figures 1 and 2. We also take the stationary axisymmetric unperturbed disk to have a purely circular velocity $v^\phi = r\Omega(r, z)$, where r is now the cylindrical radial coordinate of our orthonormal basis. The small radial velocity of actual accretion disks should not significantly affect our results, although it will modify the inner boundary condition. Thus the equations of equilibrium are simply $r\Omega^2 = \partial\Phi/\partial r + \rho^{-1}\,\partial P/\partial r$, and $\rho^{-1}\,\partial P/\partial z = -\partial\Phi/\partial z$.

The perturbations will be described by a Lagrangian displacement

$$\xi_* = \xi(r, z)e^{i(m\phi + \sigma t)} \qquad (m = 0,\ \pm 1,\ \pm 2, \ldots) . \tag{1.2}$$

In terms of the eigenfrequency σ, the corotating frequency is $\omega(r, z) = \sigma + m\Omega$. For the models which we shall consider, we shall see that typically

$$h \lesssim \lambda \ll r , \qquad \xi^z \lesssim (h/\lambda)\xi^r . \tag{1.3}$$

Here $h(r)$ is the effective half-thickness of the unperturbed disk, and the radial scale length $\lambda \equiv 2\pi\,|\,\xi^r/(\partial\xi^r/\partial r)\,|$.

2. GENERAL FORMALISM FOR ADIABATIC OSCILLATIONS

We are considering an unperturbed model for which $v^r = 0$, which implies that the accretion rate $dM/dt = 0$. This is equivalent to neglecting the explicit effects of viscous forces. These assumptions only introduce major errors in the energy generation of the unperturbed model, not in its structure. Therefore we can consider perturbations to the equations of inviscid fluid flow

$$(\partial_t + v^j\nabla_j)v^i + \rho^{-1}\nabla^i P + \nabla^i\Phi = 0 , \tag{2.1}$$

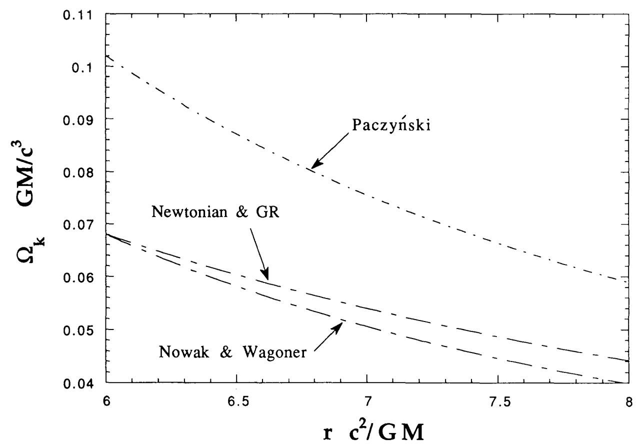

FIG. 1

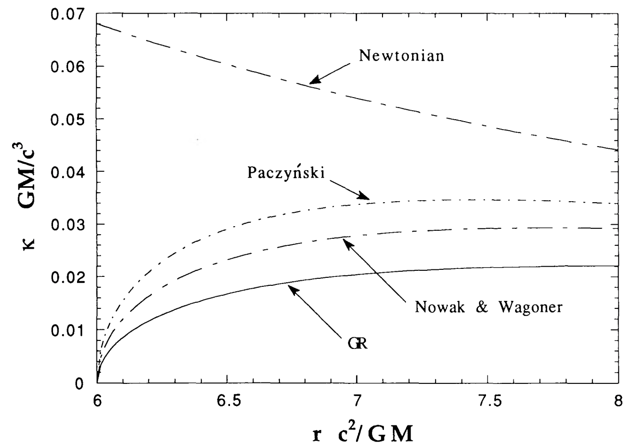

FIG. 2

FIG. 1.—Comparison of the Keplerian rotation frequency Ω_K (measured at infinity) as calculated from our potential (eq[1.1]) with the value obtained from Newtonian mechanics and general relativity (which are identical outside a nonrotating mass), and the value obtained from Paczyński's potential.

FIG. 2.—Comparison of the epicyclic frequency κ as calculated from our potential (eq[1.1]) with the value obtained from Newtonian mechanics, general relativity, and the value obtained from Paczyński's potential.

where v^i is the fluid velocity, and ρ and P are the fluid density and pressure. Following the Lagrangian formalism of Friedman & Schutz (1978a), we define a Lagrangian perturbation vector $\boldsymbol{\xi}_*$ such that the Lagrangian change of a scalar function f is given by

$$\Delta f = \delta f + \xi^i_* \nabla_i f \,, \tag{2.2}$$

and the Lagrangian change of a covariant vector field v^i is given by

$$\Delta v^i = \delta v^i + \xi^j_* \nabla_j v^i - v^j \nabla_j \xi^i_* \,, \tag{2.3}$$

where δf and δv^i are the Eulerian changes of the scalar and vector fields. This formalism is a covariant extension of the one originally developed by Lynden-Bell & Ostriker (1967).

The reason we choose to consider Lagrangian variations, rather than Eulerian ones, is that within this framework it is very simple to calculate the canonical energy of a perturbation mode (Friedman & Schutz 1978a), as well as the time variation of this energy due to various dissipative mechanisms (Friedman & Schutz 1978b). This allows us to compute growth or damping times of the modes if they are much greater than the period of oscillation. Unlike the earlier work of Lynden-Bell & Ostriker (1967), the formalism of Friedman and Schutz is applicable to the calculation of damping times of modes in disks. This is because Friedman and Schutz properly account for "trivial" perturbations, that is perturbations which relabel the fluid elements and leave the physical variables unchanged. In a future paper we will consider two dissipative mechanisms: gravitational radiation reaction force and viscous stresses.

Given the above definitions, the perturbation of the fluid velocity becomes

$$\Delta v^i = \partial_t \xi^i_* \,, \tag{2.4}$$

and the continuity equation becomes

$$\Delta \rho = -\rho \nabla_i \xi^i_* \,. \tag{2.5}$$

Since we here restrict ourselves to adiabatic perturbations, the equation of state yields

$$\frac{\Delta P}{P} = \gamma \frac{\Delta \rho}{\rho} \,, \tag{2.6}$$

where the adiabatic index $\gamma \equiv (\rho/P)(\partial P/\partial \rho)_s$. Using a semicolon to denote the covariant derivative with respect to the flat spatial metric, the equation for the adiabatic perturbations of a stationary Newtonian fluid is written as

$$\rho \partial_t^2 \xi^i_* + 2\rho v^j \partial_t \xi^i_{*;j} + (1-\gamma) P^{,i} \xi^j_{*;j} - \gamma P \xi^{j;i}_{*;j} - P_{,j} \xi^{j;i}_* - \gamma^{,i} P \xi^j_{*;j} + \rho v^j v^k_{;j} \xi^i_{*;k} + \rho v^j v^k \xi^i_{*;k;j} + \rho \Phi_{;j}^{;i} \xi^j_* = 0 \,. \tag{2.7}$$

The advantage to us of using the covariant description is that equation (2.7) does not refer to an explicit coordinate system. In this form we are free to tune our coordinates to the problem at hand. Note also that equation (2.7) is not relativistic, but rather it is Newtonian. The only post-Newtonian correction we include is to the potential (1.1).

We ignore the self-gravity of both the unperturbed disk and the perturbations. Since we are primarily concerned with thin disks we may take the radial cylindrical derivative of the spherically symmetric gravitational potential near $z = 0$ to be $\partial \Phi / \partial r = r\Omega_0^2[1 + (1/2)(z/r)^2 \partial \ln \Omega_0^2 / \partial \ln r + \mathcal{O}(z/r)^4]$, where $\Omega_0(r)$ is the Keplerian angular velocity (shown in Fig. 1). Writing equations (2.7) in the cylindrical orthonormal basis, we obtain

$$\left[\omega^2 - r\frac{\partial\Omega_0^2}{\partial r} + (\Omega^2 - \Omega_0^2)\right]\xi^r + 2i\Omega\omega\xi^\phi - z\frac{\partial\Omega_0^2}{\partial r}\xi^z + \rho^{-1}\frac{\partial P}{\partial z}\frac{\partial\xi^z}{\partial r}$$

$$+ c_s^2\frac{\partial(\nabla_*\cdot\xi)}{\partial r} + \rho^{-1}\frac{\partial P}{\partial r}\left[(\gamma-1)(\nabla_*\cdot\xi) + \frac{\partial\xi^r}{\partial r}\right] + \frac{\partial\gamma}{\partial r}\frac{P}{\rho}(\nabla_*\cdot\xi) = 0\,, \quad (2.8a)$$

$$\omega^2\xi^\phi - 2i\Omega\omega\xi^r + \frac{im}{r}\left[c_s^2(\nabla_*\cdot\xi) + \rho^{-1}\frac{\partial P}{\partial r}\xi^r + \rho^{-1}\frac{\partial P}{\partial z}\xi^z\right] = 0\,, \quad (2.8b)$$

$$(\omega^2 - \Omega_0^2)\xi^z - z\frac{\partial\Omega_0^2}{\partial r}\xi^r + \rho^{-1}\frac{\partial P}{\partial r}\frac{\partial\xi^r}{\partial z} + \rho^{-1}\frac{\partial P}{\partial z}\frac{\partial\xi^z}{\partial z} + c_s^2\frac{\partial(\nabla_*\cdot\xi)}{\partial z} + \left[\frac{(\gamma-1)}{\rho}\frac{\partial P}{\partial z} + \frac{\partial\gamma}{\partial z}\frac{P}{\rho}\right](\nabla_*\cdot\xi) = 0\,, \quad (2.8c)$$

where the speed of sound $c_s = (\gamma P/\rho)^{1/2}$, and terms of $\mathscr{O}(\Omega_0^2\xi^r z^2/r^2)$ and $\mathscr{O}(\Omega_0^2\xi^z z^2/r^2)$ have been ignored. (These terms, arising from the higher order z-dependence of the external potential can easily be included if thicker disks are considered.) In the above, and throughout the rest of the paper, when we write $\nabla_*\cdot\xi$ we do not mean the divergence of the vector ξ, but rather $(\nabla\cdot\xi_*)$ exp $[-i(m\phi+\sigma t)]$ (i.e., $r^{-1}\partial(r\xi^r)/\partial r + im\xi^\phi/r + \partial\xi^z/\partial z$).

The above equations are very general in that no assumptions have been made about the relative sizes of the terms. They govern the adiabatic perturbations of any non–self-gravitating cylindrically symmetric stationary fluid that has no radial or vertical component of velocity. If we include the full structure of the spherically symmetric external gravitational field, the equations are valid for either thick or thin accretion disk models. In this study we shall restrict ourselves to thin accretion disks. For the case of massive disks, one may have to include self-gravity terms. The formalism of Friedman & Schutz (1978a) allows their inclusion, Lynden-Bell & Ostriker (1967) derive an expression for the self-gravity term. Also note that we have not specified the external gravitational potential (aside from its spherical symmetry); we are free to choose a potential that is appropriate for the problem at hand. The only potential used in this paper, however, is given by equation (1.1).

The perturbation vector ξ_* must be subject to boundary conditions. Throughout this paper we shall use the condition that the Lagrangian variation of the pressure, $\Delta P = -\gamma P\nabla_i\xi_*^i$, vanishes at the unperturbed boundary. (Okazaki et al. [1987] and Kato [1989] use the condition that the wave energy density remain finite at the border. This is more restrictive than our [more usual] condition). Since P vanishes at the vertical boundary for the disk models considered, our requirement is that $\nabla_*\cdot\xi$ remain finite at the boundary. On the inner edge of the disk this condition is somewhat more problematical, and we shall defer discussion of it until § 5. We shall simply note that this condition only imposes a problem on the inner edge when the wave is free to propagate in that region of the disk (for many modes, the inner region of the disk is evanescent, and hence the mode has negligible amplitude near the inner edge).

3. SERIES EXPANSION AND SEPARABLE SOLUTIONS

Equations (2.8a)–(2.8c) are too difficult to solve as written; further approximations are necessary if we are to gain any insight into their general solution. We now shall restrict ourselves to the perturbations of thin disks in hydrostatic equilibrium ($\rho^{-1}\partial P/\partial z = -\Omega_0^2 z$). Let us consider perturbations for which $h \lesssim \lambda \ll r$ and $\lambda\omega \sim c_s(\sim h\Omega)$, and work to lowest WKB order in λ/r. (We take $\partial\xi^z/\partial r \sim \xi^z/\lambda$ and $\partial P/\partial r \sim P/r$, etc.) To this order $\Omega^2(r, z) = \Omega_0^2(r)$, so we shall drop the subscript, and we will also ignore the derivatives of γ. The "WKB version" of equations (2.8) is then

$$\xi^\phi = 2i(\Omega/\omega)\xi^r\,, \quad (3.1a)$$

$$(\omega^2 - \kappa^2)\xi^r + c_s^2\frac{\partial(\nabla_*\cdot\xi)}{\partial r} - \Omega^2 z\frac{\partial\xi^z}{\partial r} = 0\,, \quad (3.1b)$$

$$(\omega^2 - \Omega^2)\xi^z + c_s^2\frac{\partial(\nabla_*\cdot\xi)}{\partial z} - (\gamma-1)\Omega^2 z(\nabla_*\cdot\xi) - \Omega^2 z\frac{\partial\xi^z}{\partial z} = 0\,, \quad (3.1c)$$

where the square of the radial epicylic frequency is given by $\kappa^2 = 4\Omega^2 + r\,\partial\Omega^2/\partial r$. The above equations become invalid when ω approaches zero (the corotation resonance). In what follows we shall assume that we are far enough away from this resonance that we can ignore its effects. Later in §§ 4 and 5 we will be more specific about how we handle this resonance for the modes that we compute.

3.1. *Series Solutions*

First let us explore equations (3.1) by expanding them about $z = 0$. We use the following definitions:

$$\xi^r(r, z) = \xi_0^r(r) + \xi_1^r(r)\left(\frac{z}{h}\right) + \xi_2^r(r)\left(\frac{z}{h}\right)^2 + \cdots\,, \quad (3.2a)$$

$$\xi^z(r, z) = \xi_0^z(r) + \xi_1^z(r)\left(\frac{z}{h}\right) + \xi_2^z(r)\left(\frac{z}{h}\right)^2 + \cdots\,, \quad (3.2b)$$

$$c_s^2(r, z) = c_s^2(r, 0)[1 - f(r, z)]\,, \quad (3.2c)$$

where $c_s^2(r, 0) \equiv \gamma(r)h^2(r)\Omega^2(r)$ defines h, and we expand $f(r, z)$ as

$$f(r, z) = f_2(r)\left(\frac{z}{h}\right)^2 + f_4(r)\left(\frac{z}{h}\right)^4 + \cdots . \tag{3.2d}$$

{It can be shown that $f_2(r) = \frac{1}{2}[1 + h^2(\partial^2 \ln \rho/\partial z^2)_{z=0}]$, from hydrostatic equilibrium. For most disk models this yields values of f_2 between 0 and $\frac{1}{2}$}. We make the consistent assumption that $\partial h/\partial r \sim h/r$, and hence is negligible, so we freely commute h through any derivative. Recall that we have assumed that radial derivatives of all the variables of the unperturbed model scale as $1/r$. For brevity we find it convenient to define $\partial/\partial r_m \equiv \partial/\partial r - 2m\Omega/\omega r$, $\zeta \equiv (\gamma - 1)/\gamma$, $v_r \equiv (\omega^2 - \kappa^2)/\gamma\Omega^2$, and $v_{z(j)} \equiv [\omega^2 - (1 + j)\Omega^2]/\gamma\Omega^2$. Substituting the above definitions, the perturbation equations (3.1b)–(3.1c) can be rewritten as

$$v_r \xi_j^r - \frac{h}{\gamma}\frac{\partial \xi_{j-1}^z}{\partial r} + h\frac{\partial}{\partial r}\left[h\frac{\partial \xi_j^r}{\partial r_m} + (j+1)\xi_{j+1}^z\right] - \sum_{k=2,4,\ldots} f_k h \frac{\partial}{\partial r}\left[h\frac{\partial \xi_{j-k}^r}{\partial r_m} + (j-k+1)\xi_{j-k+1}^z\right] = 0 , \tag{3.3a}$$

$$v_{z(j-1)}\xi_{j-1}^z + j\left[h\frac{\partial \xi_j^r}{\partial r_m} + (j+1)\xi_{j+1}^z\right] - \sum_{k=2,4,\ldots}[\zeta\delta_{k2} + (j-k)f_k]\left[h\frac{\partial \xi_{j-k}^r}{\partial r_m} + (j-k+1)\xi_{j-k+1}^z\right] = 0 , \tag{3.3b}$$

where $j = 0, 1, 2, \ldots$ and $\xi_n^r = \xi_n^z = 0$ for $n < 0$.

Several things can be noticed from the above equations. Because of the even parity about $z = 0$ of the unperturbed disk, only even/odd terms of ξ^r and odd/even terms of ξ^z enter into the equations at a given order. The displacements therefore naturally subdivide into even and odd modes (denoted by whether $\nabla_* \cdot \boldsymbol{\xi}$ is even or odd about $z = 0$). Also note that if we consider the equations through order j for each parity $(-1)^j$, we have $j + 1$ equations and $j + 2$ unknowns. The extra equation is provided by the boundary conditions at $|z| \gg h$. As stated before, we take $P\nabla_* \cdot \boldsymbol{\xi}$ to vanish at the unperturbed boundary.

Our boundary condition does not allow us to terminate our series solution at finite j for all cases. If we terminate our series solution at some value J such that $\xi_j^r = \xi_{j-1}^z = 0$ for $j > J$, we can then rewrite equations (3.3) as

$$\zeta h \frac{\partial^2 \xi_J^r}{\partial r \partial r_m} = \sum_{k=2,4,\ldots} k f_{k+2} \frac{\partial}{\partial r}\left[h\frac{\partial \xi_{J-k}^r}{\partial r_m} + (J-k+1)\xi_{J-k+1}^z\right] . \tag{3.4}$$

We see that if we set $f_k = 0$ (c_s a function of r only) the finite series solution is trivial unless $\zeta = 0$. If $\zeta \neq 0$, for $f_k = 0$, then $\xi_J^r = 0$ which forces us to terminate our series at a lower order. The Lagrangian perturbation at the lower order again satisfies equation (3.4), with J replaced by $J - 2$, and hence we find ξ_{J-2}^r, etc. In this manner we are forced to set $\xi_J^r = 0$ at all orders.

We have the possibility of obtaining a finite series solution if we know f_k, which requires a specific disk model, or if $\zeta = f_k = 0$. Equation (3.4) is self-consistent for this latter case, which is the case explored by Okazaki et al. (1987) and Kato (1989). For their case, equations (3.3) then give

$$v_r v_{z(J-1)}\xi_J^r + \frac{h^2}{\gamma}\frac{\omega^2}{\Omega^2}\frac{\partial^2 \xi_J^r}{\partial r \partial r_m} = 0 . \tag{3.5}$$

This leads to the dispersion relation [for $\xi_J^r \propto \exp(ikr)$]

$$(\omega^2 - J\Omega^2)(\omega^2 - \kappa^2) = \omega^2 c_s^2 k^2 , \tag{3.6}$$

if $m\lambda^2/hr \ll 1$. This dispersion relation has the interesting property that waves are trapped in regions where $\omega^2 < \kappa^2$ and $\omega^2 < J\Omega^2$, or in regions where $\omega^2 > \kappa^2$ and $\omega^2 > J\Omega^2$. For the cases studied by Okazaki et al., the wave is evanescent near the inner edge of the disk (where $\kappa \to 0$), and is trapped at radii near that of the maximum epicyclic frequency.

We are more interested in realistic cases, however, for which ζ and f_i are not equal to zero. We can gain some insight into the general solution by considering the first few orders in the series expansion. Let us consider the two lowest even modes. We have three equations (at orders z^0, z^1, and z^2; eq. [3.3a] for $j = 0$, 2 and eq. [3.3b] for $j = 1$) and four unknowns (ξ_0^r, ξ_2^r, ξ_1^z, and ξ_3^z). In order to make the problem tractable, let us take $\xi_2^r(r) = \Upsilon(r)\xi_0^r(r)$, and let us assume that $\Upsilon(r)$ is a slowly varying function of r (i.e., $\partial\Upsilon/\partial r \sim \Upsilon/r$ and hence is negligible). This is expected since the vertical structure of the unperturbed disk should determine $\Upsilon(r)$. We then have a system of equations that we can solve. The solution for ξ_0^r is given by

$$\{\omega^2 + [(1 - \gamma) + 2\gamma(\Upsilon + f_2)]\Omega^2\}(\omega^2 - \kappa^2)\xi_0^r = -\omega^2 c_s^2(r, 0)\frac{\partial^2 \xi_0^r}{\partial r \partial r_m} . \tag{3.7}$$

We expect that $\Upsilon(r) \sim \mathcal{O}(1)$, although there is no *a priori* way of knowing its sign. It can also be shown, using this approximation, that $h^{-1}\partial\xi_1^z/\partial r \cong -\partial^2\xi_0^r/\partial r \partial r_m$. That is, $\xi_1^z \sim \mathcal{O}[(h/\lambda)\xi_0^r]$.

If we look at the odd modes through order z and assume $\xi_2^z(r) = \Gamma(r)\xi_0^z(r)$ (where Γ is again a slowly varying function of r), then we find that

$$[\omega^2 + (2\gamma\Gamma - 1)\Omega^2](\omega^2 - \kappa^2)\xi_1^r = -\omega^2 c_s^2(r, 0)\frac{\partial^2 \xi_1^r}{\partial r \partial r_m} . \tag{3.8}$$

The variation of the speed of sound with z does not explicitly come in at this order; however, it undoubtedly plays a large role in determining $\Gamma(r)$. Again it can be shown that under these assumptions $\xi_0^z \sim \mathcal{O}[(h/\lambda)\xi_1^r]$.

In general we see that the perturbations satisfy dispersion relations of the form

$$[\omega^2 + g(r)\Omega^2](\omega^2 - \kappa^2) = \omega^2 c_s^2(r, 0)k^2 , \tag{3.9}$$

where $g(r)$ is of order unity and is a slowly varying function of r. However, $g(r)$ can be of either sign, which strongly determines in which regions of the disk eigenmodes can be trapped. For positive $g(r)$, waves are propagating in regions where $\omega^2 > \kappa^2$, and evanescent in regions where $\omega^2 < \kappa^2$. This causes modes to be trapped near the inner edge of the disk where $\kappa^2 = 0$. For negative $g(r)$ it is possible to trap waves in regions where $\omega^2 < \kappa^2$, which would exclude the inner edge of the disk.

3.2. *Separable Solutions*

The work of Okazaki et al. (1987) and Kato (1989) also can be considered from the point of view of solutions that are separable into functions solely dependent upon r or z. [In practice we separate ξ^r and ξ^z into functions of the form $f(r)g(z/h)$, but since we are taking $h(r)$ to be slowly varying this is equivalent to separating in r and z.] A consequence of the assumptions that $\zeta = f_i = 0$ and that the radial derivatives of all variables, aside from the perturbation vector, scale as $1/r$ is that both ξ^r and ξ^z vary in the radial direction, to lowest order, with the *identical* wavelength. This is easily seen by looking at equations (3.3). Under the above assumptions $j(\omega^2 - \kappa^2)\xi^r_j = \omega^2 h\, \partial\xi^z_{j-1}/\partial r$. Hence if ξ^r is separable, so is ξ^z. Likewise since ω^2, κ^2, and h are all assumed to be slowly varying, ξ^r and ξ^z vary with the same wavelength. That is, we can define a radial wavenumber, $k(r)$, such that

$$k(r)^2 \equiv -\frac{1}{\xi^r}\frac{\partial^2\xi^r}{\partial r^2} = -\frac{1}{\xi^z}\frac{\partial^2\xi^z}{\partial r^2} , \tag{3.10}$$

where $k(r)$ is a slowly varying function of r [to $\mathcal{O}(1/\lambda)$; z-dependent corrections enter in at $\mathcal{O}(1/\sqrt{\lambda r})$]. We shall continue to use Kato's explicit assumption $f_j = 0$ and his implicit assumption that $\omega \gg (\lambda/r)m\Omega$ (Okazaki et al. treat the case $m = 0$, so are unaffected by this latter assumption); however, we shall relax the assumption $\zeta = 0$, but still use equation (3.10).

Equations (3.1b)–(3.1c) can be written in a more convenient form if we use the variable $\varphi \equiv \nabla_* \cdot \boldsymbol{\xi} \cong \partial\xi^r/\partial r + \partial\xi^z/\partial z$, in which case we obtain

$$(\omega^2 - \kappa^2)\left(\varphi - \frac{\partial\xi^z}{\partial z}\right) + c_s^2 \frac{\partial^2\varphi}{\partial r^2} - \Omega^2 z \frac{\partial^2\xi^z}{\partial r^2} = 0 , \tag{3.11a}$$

$$(\omega^2 - \Omega^2)\xi^z + c_s^2 \frac{\partial\varphi}{\partial z} - (\gamma - 1)\Omega^2 z\varphi - \Omega^2 z \frac{\partial\xi^z}{\partial z} = 0 . \tag{3.11b}$$

Using equation (3.10), we can easily solve equation (3.11a) for φ, finding

$$\varphi = \frac{(\omega^2 - \kappa^2)}{(\omega^2 - \kappa^2 - c_s^2 k^2)}\left[\frac{\partial}{\partial\eta} - \frac{c_s^2 k^2 \eta}{\gamma(\omega^2 - \kappa^2)}\right]\frac{\xi^z}{h} , \tag{3.12}$$

where we have defined $\eta \equiv z/h$.

The question now arises, does this lead to a separable solution? We can rewrite equation (3.12) as

$$\frac{\partial\xi^r}{\partial r} = \frac{c_s^2 k^2}{(\omega^2 - \kappa^2 - c_s^2 k^2)}\left[\frac{\partial}{\partial\eta} - (1 - \zeta)\eta\right]\frac{\xi^z}{h} . \tag{3.13}$$

For ζ = constant we see that if ξ^z is separable into $f(r)g(\eta)$, then so is $\partial\xi^r/\partial r$ and therefore ξ^r. Note, however, that φ is not separable but does vary, to lowest order, with the same z-independent wavelength as ξ^r and ξ^z. That is, $\varphi^{-1}(\partial^2\varphi/\partial r^2) = -k(r)^2$ plus z-dependent corrections of $\mathcal{O}(1/\lambda r)$. We find much to our distress, but not surprise, that in reality the horizontal and vertical directions are intimately coupled. The fact that the z-dependence comes in at the next order, and the fact that the physical variable φ (which is directly related to the perturbed density and pressure) is not separable, indicates that perhaps the most reasonable approach to solving equations (3.1) for the more general case would be a fully two-dimensional numerical code.

We shall continue as if our use of equation (3.10) were completely justified, to see how the solutions of Okazaki et al. and Kato are modified under more realistic approximations. Substituting for φ in equation (3.11b), we obtain

$$\left[\frac{\partial^2}{\partial\eta^2} - \eta\frac{\partial}{\partial\eta} + \frac{(\omega^2 - \Omega^2)(\omega^2 - \kappa^2) - \omega^2 c_s^2 k^2}{\gamma\Omega^2(\omega^2 - \kappa^2)} + \zeta(1 - \zeta)\frac{c_s^2 k^2}{(\omega^2 - \kappa^2)}\eta^2\right]\xi^z = 0 . \tag{3.14}$$

The above equation is equivalent to Kato's equations in the limit $\zeta \to 0$. (For the perturbations we wish to consider $\zeta \sim \frac{1}{4}$–$\frac{2}{5}$.)

If we define

$$A(r) = \frac{(\omega^2 - \Omega^2)(\omega^2 - \kappa^2) - \omega^2 c_s^2 k^2}{\gamma\Omega^2(\omega^2 - \kappa^2)} , \qquad B(r) = \zeta(1 - \zeta)\frac{c_s^2 k^2}{(\omega^2 - \kappa^2)} , \tag{3.15}$$

we can rewrite equation (3.14) in the form of the quantum harmonic oscillator equation by making the substitutions $\xi^z = f(r)u(\eta)v(\eta)$, with $v(\eta) = \exp(\eta^2/4)$, and $x(r, z) = (1 - 4B)^{1/4}\eta$. The resulting equation for $u(x)$ is (since $|\partial/\partial_z| \gg |\partial/\partial r|$)

$$\frac{\partial^2 u}{\partial x^2} + \left[\left(J - \frac{1}{2}\right) - \frac{x^2}{4}\right]u = 0 , \tag{3.16}$$

with the "quantum condition"

$$\frac{(A + 1/2)}{(1 - 4B)^{1/2}} = J - \frac{1}{2}, \tag{3.17}$$

where J is an integer ≥ 1. The quantum condition is imposed by our choice of boundary conditions—that the modes be even or odd about zero and that they have vanishing ΔP for $z \gg h$. With these boundary conditions, the solution is

$$\xi^z = f(r) \exp\left[\frac{(\eta^2 - x^2)}{4}\right] H_{J-1}\left(\frac{x}{\sqrt{2}}\right), \tag{3.18}$$

where $H_J(x)$ is the Jth order Hermite polynomial (as defined in Abramowitz & Stegun 1972). If $\zeta \to 0$, then $B \to 0$, $x = \eta$, and the solution is given by

$$\xi^z \propto H_{J-1}(\eta/\sqrt{2}) . \tag{3.19}$$

The quantum condition (3.17) is identical to the dispersion relation (3.6), agreeing with Kato's results.

We can explore how Kato's results are modified when $\zeta \neq 0$ by looking at the approximation $|B| \gg 1$. This approximation is self-consistent for $m > 0$. The case of $m = 0$ must be studied using the full dispersion relationship, equation (3.17). (This is because the $m > 0$ modes lie closer to $r = 6GM/c^2$, where the ratio Ω/ω is large). Under this approximation, equation (3.17) can be written as

$$c_s^2 k^2 = \frac{\zeta}{(1 - \zeta)} \left(\frac{\Omega}{\omega}\right)^4 (2J - 1)^2 (\kappa^2 - \omega^2) . \tag{3.20}$$

The z-component of the perturbation vector becomes

$$\xi^z = f(r) \exp\,(-|B|^{1/2} \eta^2/2) H_{J-1}(|B|^{1/4} \eta) , \tag{3.21}$$

where

$$B = -\zeta^2 \left(\frac{\Omega}{\omega}\right)^4 (2J - 1)^2 .$$

We can obtain an estimate of σ, and hence B, by employing WKB analysis. If we use equation (3.20) to integrate k between the classical turning points $\omega^2 = \kappa^2$, σ is found by solving the equation $\int (k/\pi) dr = (n + \frac{1}{2})$, where $n = 0, 1, 2, \ldots$. Results of calculating this integral for a particular $m = 2$ case are presented in Figure 3. We find that for this case the average value of $|B|$ is ~ 50. This certainly justifies our approximation of $|B| \gg 1$, as well as implies that the mode is strongly damped away from $z = 0$. These modes are almost inherently two-dimensional. Figure 4 gives an indication of the characteristic size of the mode, which is quite small. The WKB analysis shows that the mode is trapped within a region of $\mathcal{O}(0.1 GM/c^2)$.

It is not surprising that the more realistic modes are not as easily separable into r and z-dependences as Kato's solutions. Kato's model is equivalent to isothermal perturbations of a gas-pressure–dominated, vertically isothermal disk. Contours of constant temperature in the unperturbed model are simply cylinders of constant r. The Lagrangian variation of the temperature is zero for these $\gamma = 1$ perturbations, hence the separability of the unperturbed model is retained. Generic modes do not have a vanishing

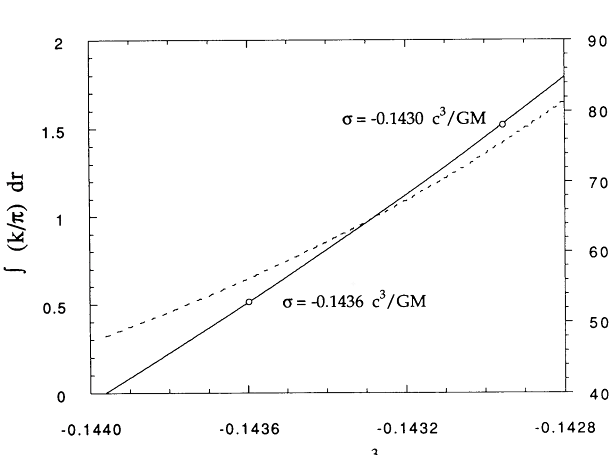

FIG. 3

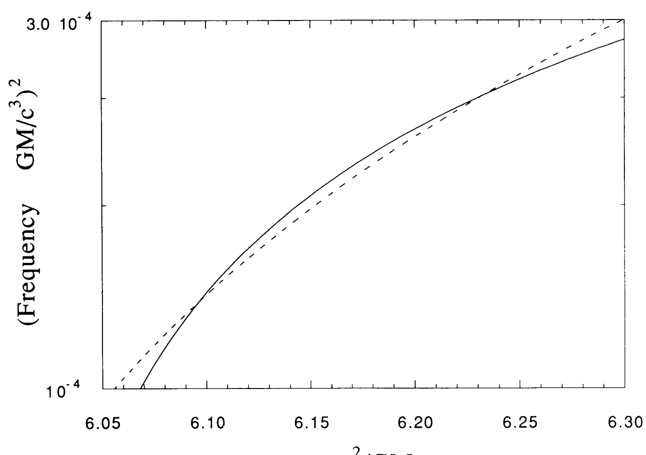

FIG. 4

FIG. 3.—Values of the WKB integral (*solid line*) of the wavenumber (normalized by dividing by π) for $m = 2$ and $J = 0$. The first two eigenvalues occur when the integral equals 1/2 and 3/2, respectively. Also plotted is the value of B averaged over the range of the integration (*dashed line*). The model parameters were $M = 10$ $M_{\odot}$, $\dot{M} = 2.5 \times 10^{17} g\, s^{-1}$, $\alpha = 10^{-3}$.

FIG. 4.—Classical turning points for the lowest WKB eigenvalue from Fig. 3. The solid line is κ^2, and the dashed line is ω^2. The mode is trapped in the region $\omega^2 < \kappa^2$.

Lagrangian variation of temperature, and as stated before the z-dependent corrections to the square of the wavenumber, $k(r)^2$, are of $\mathcal{O}(1/\lambda r)$. These effects lead to stronger coupling between the radial and vertical directions, and hence a more complicated dispersion relationship.

4. PURELY HORIZONTAL OSCILLATIONS

It turns out that for cases in which the speed of sound c_s as well as the adiabatic index γ are only functions of r we indeed can find a separable solution. Returning to equations (2.8), we can obtain a simple solution for the case of $\xi^z = 0$; that is, the solution with no motion normal to the midplane. As it also turns out, we are able to work to higher order than before, so we retain terms of $\mathcal{O}[(h^2/\lambda r)\Omega^2\xi^r]$ and greater in equations (2.8a) and (2.8b) and terms of $\mathcal{O}[(h/\lambda)\Omega^2\xi^r]$ in equation (2.8c). Once again, to this order $\Omega(r, z) = \Omega_0(r)$. The $z(h/\lambda)r^2$ component of equation (2.7) (given by eq. [2.8c]) becomes

$$\frac{\partial(\nabla_* \cdot \xi)}{\partial z} \cong \zeta \frac{z}{h^2} (\nabla_* \cdot \xi) , \tag{4.1}$$

where ζ and h are defined as before. We shall again assume that $\partial h/\partial r \sim h/r$. Equation (4.1) then gives

$$(\nabla_* \cdot \xi) = f(r)e^{\zeta(z/h)^2/2} , \tag{4.2}$$

where $f(r)$ is a function of r to be determined. To within the above approximations, we can then take the z-dependence of $\xi^r \propto e^{\zeta(z/h)^2/2}$ if $\xi^r/\lambda \gg \xi^\phi/r$. Equation (2.8b) (the ϕ component of eq. [2.7]) becomes

$$\xi^\phi = i\omega_*^{-2}\left(2\Omega\omega\xi^r - \frac{m}{r} c_s^2 \frac{\partial \xi^r}{\partial r}\right) , \tag{4.3}$$

where $\omega_*^2 = [\omega^2 - (m/r)^2 c_s^2]$. Substituting the above into equation (2.8a) (the r component) yields

$$(\omega^2 - \kappa^2)\xi^r + c_s^2\left[\frac{\partial^2 \xi^r}{\partial r^2} + r^{-1} \frac{\partial \ln(\gamma P r)}{\partial \ln r} \frac{\partial \xi^r}{\partial r}\right] = 0 , \tag{4.4}$$

if again $\lambda\omega \sim c_s(\sim h\Omega)$. This latter approximation breaks down, as noted before, near the corotation resonance, which has been shifted from $\omega \approx 0$ to $\omega \approx \pm(m/r)c_s$. This region tends to be very narrow $[\mathcal{O}(h)]$, and can be handled numerically by introducing a small imaginary component to σ $[Im(\sigma)/Re(\sigma) \sim 10^{-3}]$. The more proper equation to solve in this case is

$$c_s^2\left[1 + \left(\frac{mc_s}{r\omega_*}\right)^2\right] \frac{\partial^2 \xi^r}{\partial r^2} + \frac{c_s^2}{r}\left[\frac{\partial \ln(\gamma P r)}{\partial \ln r} + \frac{(mc_s)^2}{r} \frac{\partial \omega_*^{-2}}{\partial r}\right] \frac{\partial \xi^r}{\partial r} + \left[\omega^2 - \kappa^2 + 4\Omega^2\left(1 - \frac{\omega^2}{\omega_*^2}\right) - \frac{2mc_s^2 \omega\Omega}{r} \frac{\partial \omega_*^{-2}}{\partial r}\right]\xi^r = 0 . \tag{4.5}$$

We have performed calculations with this equation, but the results are qualitatively and nearly quantitatively the same as from equation (4.4). Throughout the rest of this work we shall only concern ourselves with solutions of equation (4.4).

If we keep only terms of $\mathcal{O}[(h/\lambda)^2\Omega^2\xi^r]$ in equation (4.4) and choose a solution proportional to exp (ikr) we obtain the standard WKB dispersion relation of disk theory (Binney & Tremaine 1987):

$$\omega^2 = \kappa^2 + c_s^2 k^2 . \tag{4.6}$$

This highly specialized result has been obtained in most previous treatments by vertically averaging the disk model before perturbing it.

The above dispersion relation implies that modes are evanescent in regions where $\omega^2 < \kappa^2$. This allows the modes to be trapped in the inner region of a disk where general relativity forces the epicyclic frequency to rise from zero at $r = 6GM/c^2$ to a maximum at $r \cong 8GM/c^2$—a feature absent in purely Newtonian disks (Kato & Fukue 1980). Making the substitution $\xi^r = (\gamma P r)^{-1/2}\Psi$ in equation (4.4), we obtain

$$\frac{\partial^2 \Psi}{\partial r^2} + V(r)\Psi = 0 , \tag{4.7a}$$

with

$$V(r) = \frac{(\omega^2 - \kappa^2)}{c_s^2} . \tag{4.7b}$$

This is the equation that we solve numerically.

For the case of $m = 0$ we can reduce equation (4.7a) to the form of the dimensionless Schrödinger equation for the eigenfrequency σ. Let us take the speed of sound to be constant in the radial direction, and let us use a new dimensionless variable $x = rc^2/GM - 6$. We can then write $V(x) \cong \Lambda - (c/c_s)^2(GM/c^3)^2\kappa^2(x)$, where $\Lambda \equiv (c/c_s)^2(GM/c^3)^2\sigma^2$. We shall concern ourselves primarily with solutions near the inner edge, so we expand $V(x)$ near $x \approx 0$ to obtain $V(x) \approx \Lambda - \mu x$ with $\mu \approx 1.5 \times 10^{-3}(c/c_s)^2$. Our equation becomes

$$\frac{\partial^2 \Psi}{\partial x^2} + (\Lambda - \mu x)\Psi = 0 , \tag{4.8}$$

which is just Airy's equation, solved by $\Psi = \text{Ai}\,[\mu^{1/3}(x - \Lambda/\mu)]$ (the z-dependence having been factored out), with Λ being determined by the boundary conditions. If we choose $\nabla_* \cdot \xi = 0$ on the inner edge (our boundary condition for disk models where P does not vanish on the inner edge of the disk), then

$$\Lambda \cong \mu^{2/3} \Rightarrow \sigma \cong 0.116\left(\frac{c_s}{c}\right)^{1/3} \frac{c^3}{GM}, \tag{4.9}$$

which qualitatively agrees with the results of Kato & Fukue (1980). To obtain $\xi^r(r)$, we simply multiply $\Psi(r)$ by $r^{-1/2}$. This is a geometrical factor which can be understood by looking at a constant pressure and density disk in the absence of gravity. If we start a wave propagating radially outward, the wavelength will remain unchanged; however, the wave amplitude must fall as $1/r^{1/2}$ in order for energy to be conserved. This behavior is accounted for in our solution.

5. NUMERICAL RESULTS

As stated before, the only equation that we solve numerically is equation (4.7). We take for our unperturbed models gas-pressure–dominated α-disks. This is done for two reasons: (1) we do not wish to address the question of stability of the unperturbed model, so we shall ignore the unstable radiation-pressure–dominated solutions, and (2) a gas pressure dominated disk is more likely to have a vertically uniform sound speed profile (Shakura & Sunyaev 1973). A gas-pressure–dominated isothermal disk in hydrostatic equilibrium also has $P \propto \exp\,[-(1/2)(z/h)^2]$, so our boundary condition of $P\nabla_* \cdot \xi = 0$ is automatically satisfied as $z \to \infty$, if $\nabla_* \cdot \xi$ remains finite. The α-disk does present problems for the boundary conditions on the inner edge of the disk. If we use the "no-torque" boundary condition, several physical variables, such as radial velocity in the case of gas-pressure–dominated disks or surface density in the case of radiation-pressure–dominated disks, become infinite. We eliminate this problem by introducing a finite torque on the inner edge. (For all the models presented here we chose a torque such that 0.1% of the specific angular momentum at the innermost orbit is dissipated by the torque.) This leaves us with a finite pressure on the inner edge of the disk, so our boundary condition becomes $\nabla_* \cdot \xi = 0$. This is equivalent to saying that outside the inner edge of the disk there exists a constant external pressure, such as a uniform radiation pressure, that keeps the disk in equilibrium. This is not a satisfying assumption, but it is a necessary consequence of our boundary conditions and our implicit assumption that $dM/dt = 0$ (cf. § 2).

In Figure 5 we present the results of calculations of two models: a disk around a 10 $M_\odot$ and a $10^8\,M_\odot$ black hole. In both cases $\alpha = 10^{-3}$ and the luminosity is 1% of the Eddington luminosity. The luminosity was chosen to be the maximum fraction of

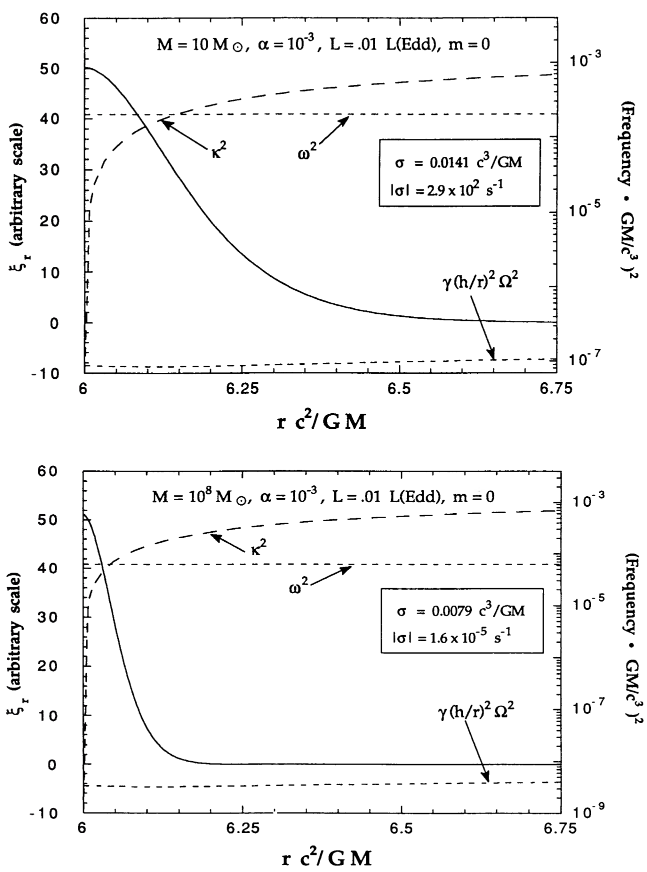

FIG. 5a

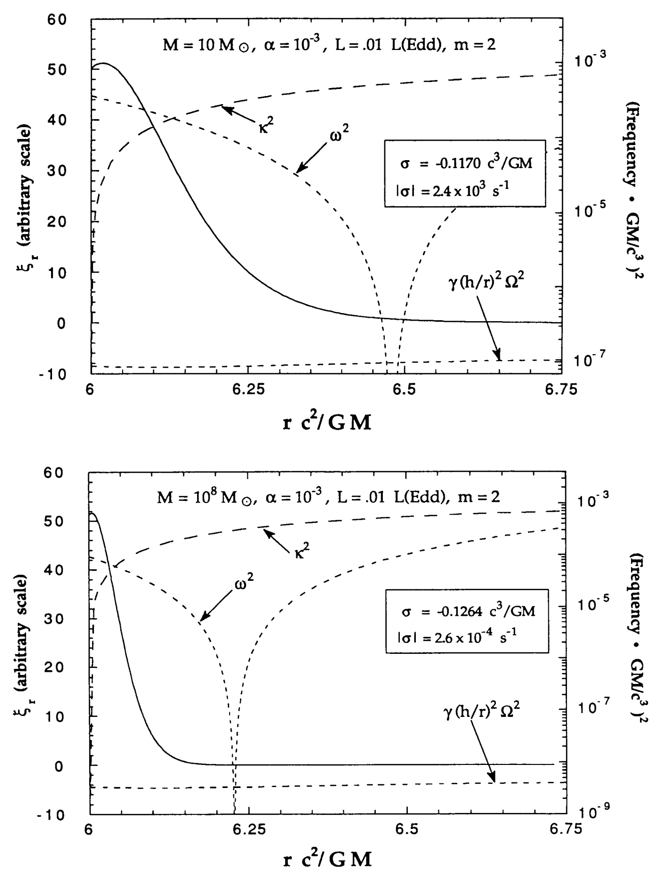

FIG. 5b

FIG. 5.—Eigenmodes for $m = 0$ (a) and $m = 2$ (b). Solid lines correspond to the radial displacement, dashed lines correspond to the local values of epicyclic and corotating frequency, as well as the value of c_s^2/r^2 (which must be $\ll \omega^2, \kappa^2$ for our approximations to be valid).

Eddington luminosity such that both disks would not have an unstable radiation pressure dominated regime. The value of α was chosen low enough such that there would be an appreciable mode growth rate (for the more massive hole) due to gravitational radiation reaction force (the CFS instability; Friedman & Schutz 1978b). This matter will be discussed fully in a future paper. We shall simply note here that $\alpha = 10^{-3}$ is not an unreasonable value (estimates range anywhere from 10^{-4} to 1). We have considered the cases of $m = 0$ and $m = 2$. The axisymmetric mode is perhaps the most readily observable as well as the lowest frequency mode, while the $m = 2$ mode is the lowest m mode strongly affected by the CFS instability.

Numerical calculations agree extremely well with the predictions of equation (4.9) (for $m = 0$). The predicted frequencies are $0.014c^3/GM$ for the 10 $M_\odot$ case and $0.008c^3/GM$ for the 10^8 $M_\odot$ case. If we replace σ by ω in equation (4.9), we obtain for the $m = 2$ modes an order of magnitude estimate for the corotating frequency on the inner edge of the disk. Numerical calculations (for $m = 2$) give the frequencies $\omega(r = 6GM/c^2) \cong 0.019c^3/GM$ for the 10 $M_\odot$ case and $\omega(r = 6GM/c^2) \cong 0.010c^3/GM$ for the 10^8 $M_\odot$ case. Note that all the modes studied are well confined to the inner edge of the disk, having wavelengths of $\mathcal{O}(0.4GM/c^2)$ $(M = 10\ M_\odot)$ and $\mathcal{O}(0.15GM/c^2)$ $(M = 10^8\ M_\odot)$. We can understand the scaling of λ if we make the estimate $\omega\lambda \sim c_s$ on the inner edge of the disk. Combining this with equation (4.9), we find that λ (in units of GM/c^2) is proportional to $(c_s/c)^{2/3}$. Measured in these units the 10 $M_\odot$ cases should have wavelengths three times larger than the 10^8 $M_\odot$ cases, which agrees with our results. Using the scaling laws we can qualitatively understand how our results will scale with such parameters as central object mass, accretion rate, and α.

We briefly note here that the results of performing calculations with an imaginary component of the frequency were qualitatively the same as the results presented in Figure 5. There was no noticeable structure at resonance for Im (σ)/Re $(\sigma) = 10^{-3}$, some structure for Im (σ)/Re $(\sigma) = 10^{-4}$, and a sharp spike comparable to the maximum mode amplitude for Im (σ)/Re $(\sigma) = 10^{-5}$. We will show in a future paper that Im (σ)/Re $(\sigma) \gtrsim 10^{-3}$ is quite reasonable for certain models of viscosity, as well as for the CFS instability of modes in disks around very massive black holes.

6. DISCUSSION AND SUMMARY

In this paper we have derived the basic Lagrangian perturbation equations governing adiabatic oscillations of thin, non–self-gravitating, accretion disks. The most general equations are accurate to any order (as long as $|\xi_*|$ is small), and include vertical oscillations as well as the vertical structure of the disk. We first explored the general case through $\mathcal{O}[(h/\lambda)^2\Omega^2\xi^r]$, via a series expansion about the midplane. This allowed us to find the general form of the dispersion relation governing oscillations in the disk. The general relation allows for modes to be trapped in one of two possible regions: the inner edge of the disk where $\kappa^2 \to 0$ or near the epicyclic frequency maximum (slightly further out). Exactly which region is the region of trapping depends upon the structure of the unperturbed disk. We then examined the possibility of solutions that the separable into functions of r and z/h. We found that such solutions are only easily found for the isothermal case ($\gamma = 1$). The solutions with more realistic values of γ have much more complicated dispersion realtionships, and are also strongly confined to the midplane for $m \neq 0$.

We then found a separable solution with no motion normal to the midplane of the disk. This solution corresponded to the standard dispersion relation of disk theory. These solutions can be well approximated analytically by Airy functions, as first pointed out by Kato & Fukue (1980). We presented numerical calculations for two central object masses, 10 $M_\odot$ and 10^8 $M_\odot$, and two different m values, $m = 0$ and $m = 2$. The frequencies of these modes were very close to the analytical predictions, and the scaling of their wavelengths agreed well with the analytical theory. These scaling laws allow us to characterize frequencies and wavelengths of disk modes by two fundamental disk parameters, the mass of the central object and the speed of sound c_s in the inner regions of the disk.

The advantage of using the Lagrangian formalism is that it also allows us to compute the rates of growth or damping of the modes due to various mechanisms. In a future paper we will consider instability due to gravitational radiation and various models of viscosity. Growth or damping due to viscous forces is dependent upon the exact form used for the viscosity law. We find that for certain cases gravitational radiation time scales may be comparable to viscous time scales for disks around supermassive black holes. The question of the importance of the self-gravity of the disk, as well as the self-gravity of the perturbations themselves, is complicated and shall be deferred to a subsequent paper. For certain ranges of parameters it is possible to find significant mode growth rate, yet still be able to ignore the effects of self-gravity. We plan to present general order of magnitude estimates for these effects in arbitrary disk models, as well as present calculations for specific modes in specific models.

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